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Full Length Research Paper

Explaining ETF Decay

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Exchange-traded funds (ETFs) have many advantages relative to mutual funds but certain types experience price decay. Price decay means that the instrument's price falls while the underlying asset's price, which it is designed to mirror, remains unchanged. Although this phenomenon is understood by sophisticated investors it is generally unknown to retail investors. The inevitability of the price decay process has never been proven mathematically. This paper provides mathematical proofs of price decay for inverse ETFs and leveraged long and short ETFs. It also identifies a way for investors to achieve inverse or leveraged returns while avoiding price decay.

Key Words Exchange traded funds, price-decay, leverage, inverse investing.

INTRODUCTION

Exchange-traded funds (ETFs) became an investment alternative in 1993. Like mutual funds, ETFs allow investors to purchase an interest in a portfolio of securities. These portfolios target specific assets such as the S&P 500 index, government bonds, commodities such as gold, and baskets of currencies. More exotic varieties of ETFs may add leverage to their portfolios (typically seeking a 2X or 3X multiple over the underlying bundle of assets) or may attempt to achieve an inverse relationship to the underlying asset's values. This article focuses on leveraged and inverse ETFs and documents why inverse funds and leveraged long ETFs experience a natural "decay" in their valuations.

ETFs have several advantages over traditional mutual funds but also suffer relative to mutual funds in other ways. The biggest advantage of ETFs is that they trade continuously during the day allowing market, limit, and short sales, while mutual funds trade only at day end, with purchases and sales occurring at the net asset value (NAV) calculated at market closing. However, ETFs do not impose short-term redemption fees, which are fairly common among mutual funds. In addition, the creativity of ETF issuers has made available pools of securities that small investors might otherwise not have access to such as commodities or currencies.

The average investor is familiar with purchasing stocks or mutual funds. Some may even have engaged in short selling. In the eyes of most investors, ETFs are like stocks except for known differences such as those articulated above. For many ETFs this conclusion is accurate but it is woefully wrong for certain classes of ETFs. In particular, differences between ETFs and stocks arise when the ETF uses leverage and when the ETF mirrors the inverse of performance.

LITERATURE REVIEW

The academic literature on ETFs is surprisingly lean. Much of this work involves comparing ETFs to traditional mutual funds. Piard (2013) does a good job of describing beta-slippage in layman's terms.

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Engle and Sarkar (2006) studied differences in the overestimation of premiums and discounts between ETFs and traditional mutual funds. Liu and Dash (2012) work is similar. Harper et al. (2006) consider how well ETFs compare to closed-end country mutual funds and find that a strategy for investing with ETFs may be superior to traditional mutual fund country funds.

The literature on ETF decay is also sparse. Carver (2009) demonstrates how as a consequence of ill-timed rebalancing and the geometric nature of return compounding highly leveraged ETFs may converge to zero in the long run; he recommends adaptive releveraging as a solution to the problem. Jarrow (2010) notes how ETF volatility causes a leveraged ETF to underperform the actual leveraged return of an index. Avellaneda and Zhang (2010) show that with dynamic rebalancing a leveraged ETF can replicate the returns of an underlying index; however, they conclude that leveraged ETFs may not be suitable for long-term investors. Each of these papers utilizes the decay properties of ETFs; none of them discusses which types of ETFs decay or whether ETFs must decay. This article addresses these two concerns explicitly.

This article accomplishes several disparate goals. First, it documents the existence of a structural problem that causes the non-symmetric relationship between long and inverse ETFs. Inverse ETFs are shown to experience natural price decay. Second, it mathematically proves that unlevered inverse ETFs must decay while unlevered long ETFs do not. In addition, it mathematically shows that with leverage both types of ETFs experience decay. Third, it suggests an alternate approach for investors to achieve similar results without experiencing decay.

METHODOLOGY

How Investors Perceive ETFs

Investors, like most individuals, make decisions based on symmetrical expectations. Zero-sum games are a perfect example of symmetry. For example, the money won by winners at a poker tournament exactly equals the money lost by losers. As another example, a winning baseball team goes one game up in the standings while the losing team falls behind by one game. Symmetry even occurs in nature; across a year every spot on earth on average has 12 hours of sunlight and 12 hours of darkness.

Symmetry is evident too in the investment world. Symmetrical relationships are arguably preferred to non symmetric ones because they are analytically easier to understand and are easier to hedge. For example, the sum of investor gains and losses equals zero, ignoring commission costs, on an equal-sized joint long (one share) short (one share) investment on a single security. Of course, un hedged long or short investments are asymmetric since they provide returns in one direction alone. A second related example of symmetry in the investment world is how an individual's gain from buying a stock is balanced by the seller's loss. This abundance of symmetry in the investment world creates a natural tendency among investors to presume that everything they encounter especially pair wise investments, is symmetric. But non symmetric investments do exist and they can catch unsophisticated investors unaware. Symmetrical expectations are helpful provided that the underlying process is symmetrical; if processes are asymmetrical the assumption may cause problems.

An example of a non symmetric pair wise investment is the purchase of both an ETF and inverse ETF pair. An investor might suppose that this paired investment has a zero sum as would a paired long/short investment. This is not true. Possibly leading the investor to reach this false conclusion is the way that the ETF and inverse ETF always move in opposite direction; however, the cumulative changes in the two securities, in most situations, are unequal. While one security rises and the other falls, an equal investment in a regular ETF balanced by an equal investment in an inverse ETF will not under most conditions, produce a zero-sum. That is, an ETF and inverse ETF pair are functionally not equivalent to a long position and short position pair and this non symmetrical characteristic of ETFs and inverse ETFs confuses investors.

There are several causes of the asymmetry: the investor (a) buys a portfolio's geometric rather than arithmetic return and (b) does not buy the underlying asset itself. In the case of unlevered ETFs, only the inverse ETF has the problem; with levered ETFs the problem afflicts both long and inverse portfolios.

Demonstration of the existence of ETF decay

Table 1: Comparison of unlevered ETF prices for two different underlying asset price trajectories.

Underlying Asset Price I.		10	9	8	10
%change			-0.100	-0.111	0.250
Time			-3	-2	-1
LONG ETF		10	9.000	8.000	10.000
SHORT ETF		10	11.000	12.222	9.167
Underlying Asset Price II.		10	11	12	10
%change			0.100	0.091	-0.167
Time			-3	-2	-1
LONG ETF		10	11.000	12.000	10.000
SHORT ETF		10	9.000	8.182	9.545

ETFs are traded in a market which establishes their price. An unlevered fund should closely track the daily return of the index or bundle of assets that the ETF represents. Generally prices track the ETFs NAV. A levered fund, for example a 2X fund, would be expected to return 200% or -200% (for an inverse fund) of the return on the assets represented by the ETF. Sponsors monitor the spread between a fund's NAV and its market price and issue or purchase ETF units to control the spread between market price and NAV.

A simple analysis demonstrates how levered and unlevered funds experience price decay. Price decay occurs because of a discrepancy between geometric and the arithmetic means. The example in Table 1 considers an unlevered ETF over three days. At the initial time, the underlying asset and both the long and short ETFs are priced at $10. Two opposing trajectories for the underlying asset price are presented: a two-day decline in price followed by a recovery at the top of the table and a two-day price increase followed by a decrease at the bottom of the table. In both cases, the underlying asset ends the three days at the same price point where it began.

The long and short (inverse) ETFs track in opposite directions the percentage daily return of the underlying asset. At the end of the three day trading period in both examples, the underlying asset price returns to its starting point $10. The long ETF also finishes the three day trading period in both cases at its original price $10. In contrast, short ETFs demonstrate decay on the third day when its closing price falls below its original $10 starting price. The short fund experiences greater than 8% decay with the price trajectory at the top of the table and more than 4% decay with the price trajectory at the bottom of the table.

With leverage, both the long and short ETFs experience price decays. This is demonstrated for 2X funds in Table 2 below with the identical two price trajectories used in Table 1. The decay process is greater with a 3X fund than for a 2X fund. The levered long fund in Table 2 experiences decays of approximately 6% with both price trajectories while the 2X short fund has decay greater than 26%at the top of the table and more than 12% decay at the bottom of the table.

Different price trajectories cause the amount of decay to vary. Generalizable results are produced with a formal mathematical treatment as in the section below. That work explores several propositions:

I. Do unlevered long ETFs experiences decay?
II. Do unlevered short ETFs experiences decay?
III. Do levered long and short ETFs experiences decay?
IV. Do either long or short levered ETFs decay by more?

Table 2: Comparison of 2X ETF prices for two different underlying asset price trajectories.

Underlying Asset Price I.		10	9	8	10
%change			-0.200	-0.222	0.500
Time			-3	-2	-1
LONG ETF		10	8.000	6.222	9.333
SHORT ETF		10	12.000	14.667	7.333
Underlying Asset Price II.		10	11	12	10
%change			0.200	0.182	-0.333
Time			-3	-2	-1
LONG ETF		10	12.000	14.182	9.455
SHORT ETF		10	8.000	6.545	8.727

MATHEMATICAL PROOFS

ETFs track the geometric return on an underlying bundle of assets. For a long ETF,

Let $ETFlong_0$ = long ETF price at time 0,

X_t be the daily return on the underlying asset on day t, and

$ETFlong_1 = ETFlong_0 * (1+ X_1)$

While for a short ETF,

Let $ETFshort_0$ = short ETF price at time 0, and

$ETFshort_1 = ETFshort_0 * (1- X_1)$

Levered ETF pricing is the same as described above except that a 2X fund replaces X_t with $2X_t$ and a 3X fund replaces X_t with $3X_t$.

Proposition I: Do unlevered long ETFs experience decay?

The first proposition asks whether long ETF's decay. Over time, an unlevered long ETF's price reflects the compound geometric return on the underlying asset. If the price of the underlying asset vacillates over time but returns to its original starting level at the end of N days, the product of unity plus the daily returns equals unity because the ETF and the underlying asset both return to their original values, as seen in equation (1).

$$\prod_1^N (1 + X_t) = 1 \qquad (1)$$

This proposition is true by definition. If an asset's price starts and ends at the same level then it must be true that the compound value of its daily returns equals zero. In that case, the product of unity plus the individual period returns equals unity since the assets price remains unchanged. Equation (1) holds for all unlevered long ETFs (and any asset for that matter) when the underlying asset vacillates daily but does not change over a specified period of time. This result proves Proposition I above that long unlevered ETFs do not decay.

A related proof shows that the sum of daily returns (X_t) cannot be negative when the underlying asset price does not change. This proof is shown below.

$$Given\ That \prod_1^N (1 + X_t) = 1, then \sum_1^N X_t \geq 0?$$

<u>Proof</u>

$$\sqrt[N]{\prod_1^N (1 + X_t)} = 1 \text{ as demonstrated by equation (1)}$$

$$And\ since\quad \frac{\sum_1^N X_t + N}{N} \geq \sqrt[N]{\prod_1^N (1+X_t)}$$

$$\Rightarrow \frac{N + \sum_1^N X_t}{N} \geq 1$$

$$\Rightarrow \frac{\sum_1^N X_t}{N} \geq 0 \Leftrightarrow \sum_1^N X_t \geq 0$$

Therefore, when an asset price begins and ends at the same level, the sum of daily returns is greater than or equal to zero.

Proposition II: Do unlevered short ETFs experience decay?

The second proposition asks whether unlevered short ETFs decay. It might be assumed by some investors that if over a period of time the underlying asset is volatile but remains unchanged and if the long ETF does not decay then the short ETF would also not decay. This is incorrect. By investing in a short/inverse ETF, investors are actually buying the inverse returns rather than the inverse price or the underlying asset. Thus a return to the original price level does not guarantee that the value of the inverse ETF investment returns to its beginning level. In fact, decay may occur, as is proven below.

$Given\ That\ \prod_1^N (1+X_t) = 1,$

then $1 = \prod_1^N (1+X_t) \geq \prod_1^N (1-X_t)$?

Proof

$$\prod_1^N (1+X_t) \geq \prod_1^N (1-X_t)$$

$$\Leftrightarrow 1 = \left[\prod_1^N (1+X_t)\right]^2 \geq \prod_1^N (1+X_t) * \prod_1^N (1-X_t) = \prod_1^N (1-X_t^2)$$

$$\Leftrightarrow 1 \geq \prod_1^N (1-X_t^2)$$

$$\Leftrightarrow X_t^2 \geq 0$$

The proposition is true since a squared value equals or exceeds zero. That is, the product of one plus the returns must be equal to or greater than the product of one minus the returns. Unlevered short ETFs decay if the underlying asset price ever changes during the investment period.

Proposition III: Do levered long and short ETFs experience decay?

This proof generalizes to cases of leverage greater than unity, i.e., nX when n = 2, 3, etc. We start, as was done above, with a related proof that is not directly a proof of decay.

a) $Given\ that\ \prod_1^N (1+X_t) = 1$, then is $\sum_1^N n * X_t \geq 0$?

Proposition I above proved that $\sum_1^N X_t \geq 0$. Therefore since N is a positive number, it must be true that $\sum_1^N n * X_t \geq 0$.

Moving on to the long decayed proof,

b) $Given\ that\ \prod_1^N (1+X_t) = 1, assuming\ X_t \neq 0,$

$$then\ \prod_1^N (1 + n * X_t) < 1?$$

Proof

$$\prod_1^N (1 + n * X_t) < 1 \Leftrightarrow \sum_1^N \ln(1 + n * X_t) < 0$$

And since we know that $\ln(1 + n * X_t) < n * \ln(1 + X_t)$ *(2)*

$$\Rightarrow \sum_1^N \ln(1 + n * X_t) < \sum_1^N n * \ln(1+X_t) = n * \ln\left[\prod_1^N (1+X_t)\right] = 0$$

Therefore, $\prod_1^N (1 + n * X_t) < 1$

this proves that long levered ETFs decay.

And the short decayed proof,

c) $Given\ that\ \prod_1^N (1+X_t) = 1, assuming\ X_t \neq 0,$

$then\ \prod_1^N (1 - n * X_t) < 1$?

Proof

$$\prod_{1}^{N}(1 - n * X_t) < 1 \Leftrightarrow \sum_{1}^{N} \ln(1 - n * X_t) < 0$$

And since we know that $\ln(1 - n * X_t) < n * \ln(1 - X_t)$ *(3)*

$$\Rightarrow \sum_{1}^{N} \ln(1 - n * X_t) < \sum_{1}^{N} n * \ln(1 - X_t) = n * \ln\left[\prod_{1}^{N}(1 - X_t)\right]$$

This holds since Proposition 2 showed that

$\prod_1^N(1 + X_t) \geq \prod_1^N(1 - X_t)$.

It is therefore true then that

$$n * \ln\left[\prod_{1}^{N}(1 - X_t)\right] < n * ln\left[\prod_{1}^{N}(1 + X_t)\right] = 0$$

Therefore $\prod_1^N(1 - n * X_t) < 1$,

this means that short leveraged ETF decay.

Proposition IV: Do either levered long or short ETFs decay by more?

$$Given\ that \prod_{1}^{N}(1 + xt) = 1, and\ assuming\ xt \neq 0,$$

$$then \prod_{1}^{N}(1 - n * xt) < \prod_{1}^{N}(1 + n * xt)?$$

Counter Example

Example 1: the long decays by more than the short.

	$10.00	$8.00	$9.00	$10.00
1X		-20.00%	12.50%	11.11%
4X		-80.00%	50.00%	44.44%
-4X		80.00%	-50.00%	-44.44%

$$\prod_{1}^{3}(1 - 4 * xt) = 50.04\% > \prod_{1}^{3}(1 + 4 * xt) = 43.32\%$$

Example 2: The short decays by more than the long.

	$10.00	$11.00	$9.90	$10.00
1X		10.00%	-10.00%	1.01%
4X		40.00%	-40.00%	4.04%
-4X		-40.00%	40.00%	-4.04%

$$\prod_{1}^{3}(1 - 4 * xt) = 80.61\% < \prod_{1}^{3}(1 + 4 * xt) = 87.39\%$$

Therefore depending on the trajectory of the underlying asset's price, either the long or short levered ETF may decay by more.

RESULTS

The demonstration tables above (in section 3) and the mathematical proofs immediately above (in section 4) clearly show that levered and inverse ETFs by their nature must decay as time progresses. This fact is escapable only by purchasing unlevered ETFs. The simplest way to invest in inverse ETFs and still not face decay is to short sell the long ETF.

In the section below we amplify strategic alternatives for investors wishing to avoid decay.

Achieving similar results without decay

Investors buying unlevered inverse funds seek an investment that moves in the opposite direction of the underlying asset. However, Proposition II above demonstrates that inverse ETFs decay. Similarly, investors buy leveraged funds in order to magnify their returns (and losses). As shown above in Proposition III, an unintended consequence of this objective is that the ETF asset suffers from decay whether or not the position is long or short as long as the underlying asset price varies.

Both objectives can be achieved without decay. An inverse investment is created by shorting the unlevered long ETF. It would be like short selling a stock and is different from investing in an inverse ETF. Here investors would be trading on the price or underlying asset rather than its return. If the asset

Table 3: Rebalancing a margined long ETF position

Day		Value	Equity	Debt
Rebalancing if Prices Fall				
0		1000	500	500
Version 1 Day 1		800	300	500
Rebalanced		600	300	300
Rebalancing if Prices Rise				
0		1000	500	500
Version 2 Day 1		1200	700	500
Rebalanced		1400	700	700

price returns to its starting level, the value of the short investment does as well, and decay does not occur under this condition.

Additional leverage can be obtained by buying the long ETF on margin. Using 50% margin would create a 2X return. It would be necessary to rebalance the margin position daily in order to maintain the 2X return. The effect of rebalancing on the investment is shown below in table 3.

Ordinary investors are unlikely to be able to or want to conduct the daily rebalancing required to maintain a non-decaying margin long ETF position as in Table 3. The resources available to investment companies allow them to offer this product to investors for a fee.

CONCLUSION

The paper proves that unlevered long ETFs do not suffer from decay and in the process of doing so shows that, if the underlying asset over a period of time returns to its original price level, the sum of the daily returns is equal to or greater than zero while the product of the geometric returns is equal to unity. In addition, the paper proves that unlevered short ETFs face decay and that levered long and short ETFs decay. Finally, the paper proves that either long or short ETF may decay more.

The paper identifies the superiority of a short position on the long ETF to an inverse ETF. Moreover, leverage can be achieved without encountering decay on a long ETF or on a short position on a long TF by buying the positions on margin. The necessary rebalancing of margin to achieve equivalence with the short ETF may be beyond the capability or time availability for small investors suggesting that investment companies might step in to fill this role.

Price decay necessarily affects levered long ETFs and levered and unlevered inverse ETFs. This decay results from the fact that the investor is buying the returns (or its inverse) on a portfolio of assets and is not buying the assets themselves. The presence of price decay persuasively argues that small investors should avoid the effected positions.

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